Inequalities 2

1. Solve $\frac{x^3-3x^2+x+1}{x^3+2x^2+3x+2} \le 0$

$\frac{(x-1)(x^2-2x-1)}{(x+1)(x^2+x+2)} \le 0$

Since $x^2+x+2 = \left(x+\frac{1}{2}\right)^2 + \frac{7}{4} > 0$ for all $x \in \mathbf{R}$

The given inequality is reduced to $\frac{(x-1)(x^2-2x-1)}{x+1} \le 0$

$(x+1)^2 \frac{(x-1)(x^2-2x-1)}{x+1} \le 0$ and $x \ne 1$

Therefore for $x \ne 1$,

$$(x+1)(x-1)(x^2-2x-1) \le 0$$

$\begin{cases} (x+1)(x-1) \le 0 \\ [x-(1-\sqrt{2})][x-(1+\sqrt{2})] \ge 0 \end{cases}$ or $\begin{cases} (x+1)(x-1) \ge 0 \\ [x-(1-\sqrt{2})][x-(1+\sqrt{2})] \le 0 \end{cases}$

$\begin{cases} -1 \le x \le 1 \\ 1-\sqrt{2} \ge x \text{ or } x \ge 1+\sqrt{2} \end{cases}$ or $\begin{cases} -1 \ge x \text{ or } x \ge 1 \\ 1-\sqrt{2} \le x \le 1+\sqrt{2} \end{cases}$ $\quad -1 \le x \le 1-\sqrt{2}$

or $1 \le x \le 1+\sqrt{2}$

Since $x \ne 1$, the complete solution is

$-1 < x \le 1-\sqrt{2}$ or $1 \le x \le 1+\sqrt{2}$.

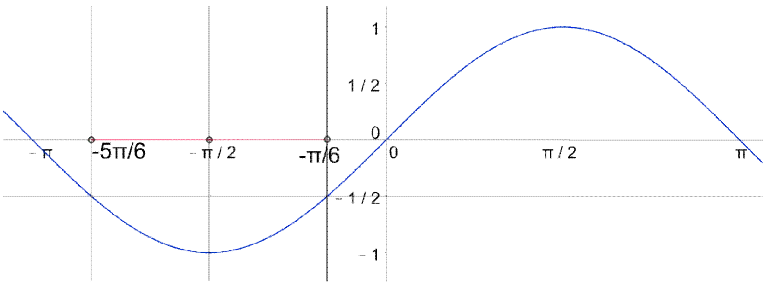
2. Solve $\cos 2\theta > 3\sin\theta + 2$ for θ, where $-\pi < \theta < \pi$.

$1 - 2\sin^2\theta > 3\sin\theta + 2$

$2\sin^2\theta + 3\sin\theta + 1 < 0$

$(\sin\theta + 1)(2\sin\theta + 1) < 0$

$-1 < \sin\theta < -\frac{1}{2}$



If $-\pi < \theta < \pi$,

the roots of $\sin\theta = -1$ is $\theta = -\frac{\pi}{2}$

and $\sin\theta = -\frac{1}{2}$ are $\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}$

$\therefore$ The solution is $-\frac{5\pi}{6} < \theta < -\frac{\pi}{2}$ or $-\frac{\pi}{2} < \theta < -\frac{\pi}{6}$.

3. If $a, b, c \geq 0$, use A.M. $\geq$ G.M., or otherwise, show that

$$\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} \geq \frac{3}{2} \ .$$

Method 1

$$\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} = \left(\frac{a+b+c}{a+b} - 1\right) + \left(\frac{a+b+c}{b+c} - 1\right) + \left(\frac{a+b+c}{c+a} - 1\right)$$

$$= \frac{a+b+c}{a+b} + \frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} - 3 = (a+b+c)\left[\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right] = 3$$

$$\geq \frac{3(a+b+c)}{[(a+b)(b+c)(c+a)]^3} - 3 \qquad \text{(A.M. } \geq \text{ G.M)}$$

$$\geq \frac{3(a+b+c)}{\frac{1}{3}[(a+b)+(b+c)+(c+a)]} - 3 \qquad \text{(A.M. } \geq \text{ G.M)}$$

$$= \frac{3(a+b+c)}{\frac{2}{3}(a+b+c)} - 3 = \frac{3}{2}$$

Method 2

By CBS inequality (Cauchy – Bunyakovskii – Schwarz inequality)

$$[(a+b)+(b+c)+(c+a)]\left[\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right] \geq (1+1+1)^2$$

$$2(a+b+c)\left[\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right] \geq 9$$

$$\frac{a+b+c}{a+b} + \frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} \geq \frac{9}{2}$$

$$\frac{c}{a+b} + 1 + \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 \geq \frac{9}{2}$$

$$\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} \geq \frac{3}{2}$$

Equality holds $\Leftrightarrow \frac{a+b}{\frac{1}{a+b}} = \frac{b+c}{\frac{1}{b+c}} = \frac{c+a}{\frac{1}{c+a}} \Leftrightarrow (a+b)^2 = (b+c)^2 = (c+a)^2 \Leftrightarrow a = b = c$

(Given : $a, b, c \geq 0$)

4. If $a, b, c \in \mathbf{R}$ and $a + b + c = 2$, show that $a^2 + b^2 + c^2 \geq \frac{4}{3}$.

Find the condition for the equality.

Method 1

Use CBS inequality (Cauchy – Bunyakovskii – Schwarz inequality) for the set

$a, b, c \ ; \ 1,1,1$

$$(a^2+b^2+c^2)(1^2+1^2+1^2) \geq (a+b+c)^2 = 2^2 = 4$$

Therefore, $a^2+b^2+c^2 \geq \frac{4}{3}$.

Equality holds $\Leftrightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1} \Leftrightarrow a = b = c = \frac{2}{3}$

Method 2

$a+b+c = 2$

$a^2+b^2+c^2+2ab+2bc+2ca = 4$

$$\begin{aligned} 3(a^2+b^2+c^2) &= 4+(a-b)^2+(b-c)^2+(c-a)^2 \\ &\geq 4 \end{aligned}$$

$a^2+b^2+c^2 \geq \frac{4}{3}$ Equality holds $\Leftrightarrow a = b = c = \frac{2}{3}$

5. Given that a, b, c, d are real numbers and $\begin{cases} a+b+c+d = 6 \\ a^2+b^2+c^2+d^2 = 12 \end{cases}$

find the maximum value of d .

Method 1

By CBS inequality, $(1^2+1^2+1^2)(a^2+b^2+c^2) \geq (a+b+c)^2$

$$3(12-d^2) \geq (6-d)^2$$
$$36-3d^2 \geq 36-12d+d^2$$
$$4d^2-12d \leq 0$$
$$d(d-3) \leq 0$$

$0 \leq d \leq 3$, hence the maximum of d is 3.

Method 2

$6-d = a+b+c$

$$\begin{aligned} (6-d)^2 &= a^2+b^2+c^2+2ab+2bc+2ca \\ &= 3(a^2+b^2+c^2)-(a-b)^2-(b-c)^2-(c-a)^2 \leq 3(a^2+b^2+c^2) \\ &= 3(12-d^2) \end{aligned}$$

Hence, $3(12-d^2) \geq (6-d)^2$

$$36-3d^2 \geq 36-12d+d^2$$
$$4d^2-12d \leq 0$$
$$d(d-3) \leq 0$$

$0 \leq d \leq 3$, hence the maximum of d is 3.

6. Solve $\left|\frac{4}{x-1}\right| \geq 3\left(1-\frac{1}{x}\right) = 3\left(\frac{x-1}{x}\right)$

Obviously, $x \neq 1$ or 0.

(a) If $x > 1$, then

$$\frac{4}{x-1} \geq 3\left(\frac{x-1}{x}\right)$$

$$\frac{4}{x-1} - 3\left(\frac{x-1}{x}\right) \geq 0$$

$$\frac{4x-3(x-1)^2}{x(x-1)} \geq 0$$

$$\frac{-3x^2+10x-3}{x(x-1)} \geq 0$$

$$\frac{3x^2-10x+3}{x(x-1)} \leq 0$$

$$\frac{(3x-1)(x-3)}{x(x-1)} \leq 0$$

$0 < x \leq \frac{1}{3}$ or $1 < x \leq 3$

But $x > 1$, therefore $1 < x \leq 3$

(b) If $x < 1$, then

$$-\frac{4}{x-1} \geq 3\left(\frac{x-1}{x}\right)$$

$$3\left(\frac{x-1}{x}\right) + \frac{4}{x-1} \leq 0$$

$$\frac{3(x-1)^2+4x}{x(x-1)} \leq 0$$

$$\frac{3x^2-2x+3}{x(x-1)} \leq 0$$

For $3x^2 - 2x + 3$, $\Delta \leq 0$

Therefore, $3x^2 - 2x + 3 \geq 0$ for all x.

The inequality become $\frac{1}{x(x-1)} \leq 0$

Solving, $0 \leq x \leq 1$

But $x < 1$ *and* $x \neq 0$,

Therefore $0 < x < 1$.

Joining (a) and (b), $\mathbf{0 < x < 1}$ *or* $\mathbf{1 < x \leq 3}$.

Yue Kwok Choy

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